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Research Article

A Note about Certain Arbitrariness in the Solution of the Homological Equation in Deprit's Method

Juan Félix San-Juan,[1] Rosario López,[2] Iván Pérez,[1] and Montserrat San-Martín[1]

[1] *Scientific Computing Group (GRUCACI), University of La Rioja, 26004 Logroño, Spain*
[2] *Center for Biomedical Research of La Rioja (CIBIR) and Scientific Computing Group (GRUCACI), 26006 Logroño, Spain*

Correspondence should be addressed to Juan Félix San-Juan; juanfelix.sanjuan@unirioja.es

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Deprit's method has been revisited in order to take advantage of certain arbitrariness arising when the inverse of the Lie operator is applied to obtain the generating function of the Lie transform. This arbitrariness is intrinsic to all perturbation techniques and can be used to demonstrate the equivalence among different perturbation methods, to remove terms from the generating function of the Lie transform, or to eliminate several angles simultaneously in the case of having a degenerate Hamiltonian.

1. Introduction

Perturbation methods represent one of the most important and powerful tools for the study of dynamical systems. These techniques are frequently used in several fields of nonlinear mechanics. The method of the averaging, which was rigorously formulated in [1], and the methods based on canonical transformations, such as the methods of von Zeipel [2, 3], Hori [4, 5], and Deprit [6], are some of the analytical perturbation methods derived from the work of Poincaré [7]. The demonstration of the equivalence of these methods [8–12] is based on certain arbitrariness closely related to the averaged equations. Moreover, this arbitrariness can also be included during the process of obtaining the generating function and determining the new system of differential equations [10, 13–16].

The purpose of this paper is to review the algorithm proposed by Deprit to perform a Lie transform so as to identify where this arbitrariness appears, how and under what conditions this can be used, and, finally, what is its relation to another method derived from the one proposed by Deprit, such as the so-called double normalization [17–19]. In order to do that, we will reexamine the processes carried out by Deprit's method when it is applied to the normalization of Hamiltonian systems of the form

$$\mathscr{H} = \mathscr{H}_0(I_1) + \sum_{n\geq 1} \frac{\varepsilon^n}{n!} \mathscr{H}_n(\mathbf{I}, \boldsymbol{\varphi}), \tag{1}$$

where ε is a small parameter and $\mathbf{I} = (I_1, \ldots, I_m)$, $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_m)$ are action-angle variables. More precisely, we will assume that the unperturbed part of (1) can be expressed in the simple form $\mathscr{H}_0 = \omega I_1$, which is what Arnold characterizes as proper degeneracy [20], while the perturbed part can be represented as a trigonometric series in the form

$$\mathscr{H}_n = \sum_{j_1,\ldots,j_m \in \mathscr{Z}} \mathscr{C}_{j_1,\ldots,j_m} \begin{pmatrix} \sin \\ \cos \end{pmatrix} (j_1\varphi_1 + \cdots + j_m\varphi_m), \tag{2}$$

where $\mathscr{C}_{j_1,\ldots,j_m}$ are functions depending on physical parameters and the actions $I_1, \ldots, I_m$. It is worth noting that all dynamical systems defined by a Hamiltonian function whose unperturbed part $\mathscr{H}_0$ is made of a finite number of harmonic oscillators can be brought to the form (1) by means of a suitable change of variables [21, 22]. Moreover, this type of Hamiltonian frequently appears in a large class of problems in celestial mechanics and classical mechanics.

For a Hamiltonian of the form (1), Deprit's method allows one to obtain the transformed Hamiltonian and the generating function of the Lie transform, order by order, as a solution of a partial differential equation. The solution of this equation is obtained by choosing the transformed Hamiltonian in the null space and the generating function in the range of the Lie operator associated to $\mathscr{H}_0$. Therefore, the transformed Hamiltonian does not depend on the angular variable φ_1 and the generating function is obtained by computing the inverse of the Lie operator. This is a procedure that usually involves integration. This fact implies that a function $\mathscr{F}_n$, belonging to the null space of the Lie operator, can be added to the generating function at each order. Usually, these functions are chosen to be equal to zero. This inherent arbitrariness introduced by this method is analogous to the gauge freedom [23–25]. It is worth noting that Morrison [10], using the von Zeipel method, considers a generating function $S_1 = \widehat{S}_1 + S_1^*$, where S_1^* is an arbitrary 2π-periodic function defined as the average of S_1 with respect to the fast variable. The determination of S_1^* can be made at the same order or postponed until next order, in which case, under certain conditions, it is possible to consider S_1^* as the generating function of a new transformation, which can be used to remove other angular variables. Another example of using Deprit's method with a zero-average generator can be seen in Metris and Exertier [26].

In Section 2, we make a brief review of some basic elements of the Lie transform perturbation theory and identify where the arbitrary function appears in Deprit's method. The process whereby Deprit's method is applied, and how explicit analytical solutions can be obtained when the arbitrary function is considered nonnull, is outlined in Section 3. The special case of degenerated first-order normalized Hamiltonians and their relation with other method derived from Deprit's method is presented in Section 4. Finally, in Section 5, an example of the normalization, when the arbitrary function is nonnull, is presented.

2. Lie Transforms

Let $\mathscr{P}$ be a Poisson algebra of functions, that is, an algebra of real or complex value functions in $(\mathbf{x}, \mathbf{X})$, where $(\mathbf{x}, \mathbf{X})$ belongs to $\mathbf{R}^n \times \mathbf{R}^n$ or $\mathbf{C}^n \times \mathbf{C}^n$, and such that for any f and g in $\mathscr{P}$ the Poisson bracket, defined by

$$\{f, g\} = \nabla_{\mathbf{x}} f \cdot \nabla_{\mathbf{X}} g - \nabla_{\mathbf{X}} f \cdot \nabla_{\mathbf{x}} g, \tag{3}$$

also belongs to $\mathscr{P}$.

A Lie transform is a uniparametric family of mappings $\phi : (\mathbf{x}', \mathbf{X}', \varepsilon) \rightarrow (\mathbf{x}, \mathbf{X})$ defined by a solution of the system of differential equations,

$$\begin{aligned} \frac{d\mathbf{x}}{d\varepsilon} &= \nabla_{\mathbf{X}} \mathscr{W}(\mathbf{x}, \mathbf{X}, \varepsilon), \\ \frac{d\mathbf{X}}{d\varepsilon} &= -\nabla_{\mathbf{x}} \mathscr{W}(\mathbf{x}, \mathbf{X}, \varepsilon), \end{aligned} \tag{4}$$

that satisfies the initial conditions $\mathbf{x}(\mathbf{x}', \mathbf{X}', 0) = \mathbf{x}'$ and $\mathbf{X}(\mathbf{x}', \mathbf{X}', 0) = \mathbf{X}'$. The function $\mathscr{W}$ is the generator of ϕ and can be expressed as the following power series in the parameter ε:

$$\mathscr{W}(\mathbf{x}, \mathbf{X}, \varepsilon) = \sum_{i \geq 0} \frac{\varepsilon^i}{i!} W_{i+1}(\mathbf{x}, \mathbf{X}). \tag{5}$$

In the Hamiltonian case, perturbation theories are based on transforming the analytical Hamiltonian function,

$$\mathscr{H}(\mathbf{x}, \mathbf{X}, \varepsilon) = \sum_{n \geq 1} \frac{\varepsilon^n}{n!} \mathscr{H}_n(\mathbf{x}, \mathbf{X}) \equiv \sum_{n \geq 1} \frac{\varepsilon^n}{n!} \mathscr{H}_{n,0}(\mathbf{x}, \mathbf{X}), \tag{6}$$

into the new one

$$\mathscr{K}\left(\mathbf{x}', \mathbf{X}', \varepsilon\right) = \sum_{n \geq 1} \frac{\varepsilon^n}{n!} \mathscr{K}_n\left(\mathbf{x}', \mathbf{X}'\right) \equiv \sum_{n \geq 1} \frac{\varepsilon^n}{n!} \mathscr{H}_{0,n}\left(\mathbf{x}', \mathbf{X}'\right), \tag{7}$$

which satisfies some specific prerequisites. One of the most useful methods to build this transform was proposed by Deprit in [6].

The Lie-Deprit method looks for a generating function $\mathscr{W}$ of ϕ so that the terms $\mathscr{H}_n$, $\mathscr{K}_n$, and $\mathscr{W}_n$ satisfy the partial differential equation, called Homological equation:

$$\mathscr{L}_{\mathscr{H}_0}(\mathscr{W}_n) + \mathscr{K}_n = \widetilde{\mathscr{H}}_{0,n}, \tag{8}$$

where $\mathscr{L}_{\mathscr{H}_0}$ is the Lie operator (or derivative) associated to $\mathscr{H}_0$, a linear operator given in terms of a Poisson bracket by

$$\mathscr{L}_{\mathscr{H}_0} : F \longrightarrow \{F, \mathscr{H}_0\}. \tag{9}$$

The right-hand side of (8), $\widetilde{\mathscr{H}}_{0,n}$, is computed from $\mathscr{H}_n$, $(\mathscr{W}_i)_{1 \leq i \leq n-1}$ and $(\mathscr{H}_{p,q})_{p+q \leq n-1}$, where the latter are obtained by means of the recursive formula:

$$\mathscr{H}_{i,j} = \mathscr{H}_{i+1,j-1} + \sum_{k=0}^{i} \binom{i}{k} \left\{\mathscr{H}_{i-k,j-1}, \mathscr{W}_{k+1}\right\}, \tag{10}$$

with $i \geq 0$ and $j \geq 0$ (for more details, see [6]).

In the case of the Hamiltonian (1), the algebra $\mathscr{P}$ can be decomposed into the direct sum

$$\mathscr{P} = \ker\left(\mathscr{L}_{\mathscr{H}_0}\right) \oplus \operatorname{im}\left(\mathscr{L}_{\mathscr{H}_0}\right), \tag{11}$$

where $\ker(\mathscr{L}_{\mathscr{H}_0})$ denotes the null space (or kernel) of $\mathscr{L}_{\mathscr{H}_0}$ and $\operatorname{im}(\mathscr{L}_{\mathscr{H}_0})$ the range of $\mathscr{L}_{\mathscr{H}_0}$.

By (11), $\widetilde{\mathscr{H}}_{0,n}$ can be uniquely split into

$$\widetilde{\mathscr{H}}_{0,n} = \widetilde{\mathscr{H}}_{0,n}^* + \widetilde{\mathscr{H}}_{0,n}^{**}, \tag{12}$$

where $\widetilde{\mathscr{H}}_{0,n}^*$ belongs to the null space of $\mathscr{L}_{\mathscr{H}_0}$ and $\widetilde{\mathscr{H}}_{0,n}^{**}$ to the range of $\mathscr{L}_{\mathscr{H}_0}$. According to (12), a solution of the Homological equation (8) is given when the new Hamiltonian is chosen to be

$$\mathscr{K}_n = \widetilde{\mathscr{H}}_{0,n}^*. \tag{13}$$

Then the generating function, $\mathscr{W}_n$, has to satisfy the identity

$$\mathscr{L}_{\mathscr{H}_0}(\mathscr{W}_n) = \widetilde{\mathscr{H}}_{0,n}^{**}. \tag{14}$$

Finally, $\mathscr{W}_n$ is obtained by solving (14). This solution is not uniquely determined by (14) since the addition of any function $\mathscr{F}_n \in \ker(\mathscr{L}_{\mathscr{H}_0})$ to $\mathscr{W}_n$ is also a solution to the above equation. Usually, a natural choice for $\mathscr{F}_n$ would seem to be to take it identically zero, although this does not necessarily have to be the most convenient choice. On the other hand, in the case of $\mathscr{F}_n \neq 0$ there are two possible instances in which the arbitrary function can be determined: first, at the same order which would produce an effect on the term $\mathscr{W}_n$ of the generating function and, second, if the determination of the arbitrary functions is postponed to the next order, then the effect may be used to express the term $\mathscr{K}_l$ with $l > n$ of the transformed Hamiltonian in a much simpler form compared to the transformed Hamiltonian obtained when $\mathscr{F}_n = 0$ is considered.

3. Normalization by the Lie-Deprit Method: Case $\mathscr{F}_n \neq 0$

Let $\mathscr{H}$ be a Hamiltonian of type (1). $\mathscr{K}$ is the normalized Hamiltonian of $\mathscr{H}$ in the phase space $(\boldsymbol{\varphi}, \mathbf{I})$ up to order n if the zero order term $\mathscr{H}_0$ is an integral of the transformed Hamiltonian; that is,

$$\mathscr{L}_{\mathscr{H}_0}(\mathscr{K}_i) = 0, \quad \text{for } i = 0, \ldots, n, \tag{15}$$

where the associated Lie operator takes the form

$$\mathscr{L}_{\mathscr{H}_0}() = \frac{1}{\omega} \frac{\partial}{\partial \varphi_1'}. \tag{16}$$

In this case, the solution to (8) is obtained when the new Hamiltonian, $\mathscr{K}_n$, is

$$\mathscr{K}_n = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\mathscr{H}}_{0n} d\varphi_1', \tag{17}$$

and the generating function, $\mathscr{W}_n$, is given by

$$\mathscr{W}_n = \frac{1}{\omega} \int \left(\widetilde{\mathscr{H}}_{0n} - \mathscr{K}_n \right) d\varphi_1' + \mathscr{F}_n, \tag{18}$$

where $\mathscr{F}_n$ is an arbitrary integration function in the null space of $\mathscr{L}_{\mathscr{H}_0}$. It is worth noting that $\mathscr{F}_n$ depends on the variables $\varphi_2', \ldots, \varphi_m'$ and all momenta I_i'. We will henceforth drop the primes from the new variables and momenta to simplify notation. For a better understanding of the role played by the arbitrary integration function $\mathscr{F}_n$, Deprit's method will be discussed taking into account its explicit appearance.

The method starts by taking $\mathscr{K}_0 = \mathscr{H}_0$. According to the Deprit's algorithm [6], at the first order the Homological equation, which has to be solved, is

$$\mathscr{L}_{\mathscr{H}_0}(\mathscr{W}_1) + \mathscr{K}_1 = \widetilde{\mathscr{H}}_{01}, \tag{19}$$

where the right-hand side of this equation is

$$\widetilde{\mathscr{H}}_{01} = \mathscr{H}_{10} = \mathscr{H}_1. \tag{20}$$

Then, by applying (17) and (18), the first-order term of $\mathscr{K}$ and $\mathscr{W}$ can be written, respectively, as

$$\mathscr{K}_1 = \frac{1}{2\pi} \int_0^{2\pi} \mathscr{H}_1 d\varphi_1, \tag{21}$$

$$\mathscr{W}_1 = \frac{1}{\omega} \int \left(\mathscr{H}_1 - \mathscr{K}_1 \right) d\varphi_1 + \mathscr{F}_1, \tag{22}$$

where $\mathscr{F}_1 \in \ker(\mathscr{L}_{\mathscr{H}_0})$. The arbitrary function can be determined at the same order; for example, taking $\mathscr{F}_1$ as

$$\mathscr{F}_1 = \frac{1}{2\pi} \int_0^{2\pi} \mathscr{W}_1 d\varphi_1, \tag{23}$$

the arbitrary function removes those terms that do not depend on φ_1 from $\mathscr{W}_1$ or postponed its determination to the next order. Now we analyze the case when the determination of $\mathscr{F}_1$ is retained. Then, at second order the right-hand side of (8) is

$$\widetilde{\mathscr{H}}_{02} = \mathscr{H}_{20} + 2\{\mathscr{H}_{10}, \mathscr{W}_1\}. \tag{24}$$

By using (20) and (22), and after some rearrangements, (24) can be written in the form

$$\widetilde{\mathscr{H}}_{02} = \widetilde{\mathscr{H}}_{02}^{\#} + 2\{\mathscr{H}_1, \mathscr{F}_1\}, \tag{25}$$

where $\widetilde{\mathscr{H}}_{02}^{\#}$ collects all the terms that do not depend on $\mathscr{F}_1$ and is given by

$$\widetilde{\mathscr{H}}_{02}^{\#} = \mathscr{H}_{20} + \frac{2}{\omega} \left\{ \mathscr{H}_{10}, \int \left(\widetilde{\mathscr{H}}_{01} - \mathscr{K}_1 \right) d\varphi_1 \right\}. \tag{26}$$

On the other hand, the second part $\{\mathscr{H}_1, \mathscr{F}_1\}$ involves the terms that only depend on the integration constant $\mathscr{F}_1$. It is worth noting that this Poisson bracket can be written as a function of the Lie operator, $-\mathscr{L}_{\mathscr{H}_1}(\mathscr{F}_1)$.

By making use of (25), the Homological equation yields

$$\mathscr{L}_{\mathscr{H}_0}(\mathscr{W}_2) + \mathscr{K}_2 = \widetilde{\mathscr{H}}_{02}^{\#} + 2\{\mathscr{H}_1, \mathscr{F}_1\}. \tag{27}$$

Now, by expanding the Poisson bracket, (27) can be rewritten as

$$\mathscr{L}_{\mathscr{H}_0}(\mathscr{W}_2) + \mathscr{K}_2 = \widetilde{\mathscr{H}}_{02}^{\#} + 2\sum_{i=2}^{m} \left(\frac{\partial \mathscr{H}_1}{\partial \varphi_i} \frac{\partial \mathscr{F}_1}{\partial I_i} - \frac{\partial \mathscr{H}_1}{\partial I_i} \frac{\partial \mathscr{F}_1}{\partial \varphi_i} \right). \tag{28}$$

Then, by applying (17) and taking into account (28), we obtain $\mathscr{K}_2$ as

$$\mathscr{K}_2 = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\mathscr{H}}_{02}^{\#} d\varphi_1 + 2\sum_{i=2}^{m} \left(\frac{\partial \mathscr{K}_1}{\partial \varphi_i} \frac{\partial \mathscr{F}_1}{\partial I_i} - \frac{\partial \mathscr{K}_1}{\partial I_i} \frac{\partial \mathscr{F}_1}{\partial \varphi_i} \right). \tag{29}$$

It is worth noting that the dependency of $\mathscr{F}_1$ on $\mathscr{K}_2$ still remains through its partial derivatives with respect to φ_i (with $i = 2, \ldots, m$) and all momenta. The elimination of this dependency is used to express $\mathscr{K}_2$ in a simpler form; it is the fundamental idea that can be found behind the explicit manipulation of the arbitrary integration function.

Therefore, according to this purpose, the first term of (29) is decomposed as the sum $\mathscr{K}_2^I + \mathscr{K}_2^{II}$, where $\mathscr{K}_2^{II}$ is to be made of those terms that we want to remove. Finally, $\mathscr{F}_1$ can be determined by solving

$$\sum_{i=2}^{m}\left(\frac{\partial \mathscr{K}_1}{\partial \varphi_i}\frac{\partial \mathscr{F}_1}{\partial I_i} - \frac{\partial \mathscr{K}_1}{\partial I_i}\frac{\partial \mathscr{F}_1}{\partial \varphi_i}\right) = \frac{1}{2}\mathscr{K}_2^{II}. \tag{30}$$

It is worth noting that the complexity of this identity directly depends on the form of $\mathscr{K}_1$ and, therefore, on the form of $\mathscr{H}_1$. Finally, the second-order term $\mathscr{K}_2$ of the Hamiltonian $\mathscr{K}$ is taken as

$$\mathscr{K}_2 = \mathscr{K}_2^I. \tag{31}$$

Now, by substituting the value of $\mathscr{F}_1$ into (22) in order to complete $\mathscr{W}_1$ and into the previous terms $\mathscr{H}_{i\,j}$ of (10), the second-order term $\mathscr{W}_2$ of $\mathscr{W}$ is

$$\mathscr{W}_2 = \frac{1}{\omega}\int\left(\widetilde{\mathscr{H}}_{0\,2} - \mathscr{K}_2\right)d\varphi_1 + \mathscr{F}_2. \tag{32}$$

As already mentioned before, the arbitrary function can be determined at the same order so as to remove those terms that do not depend on φ_1 from $\mathscr{W}_1$ or postpone its determination to the next order.

The above reasoning can be immediately extended to order n in exactly the same way. By (10), the right-hand side of (8) can be written in the following generic form:

$$\widetilde{\mathscr{H}}_{0\,n} = \widetilde{\mathscr{H}}_{0\,n}^{\#} + n\left\{\mathscr{H}_{1\,0}, \mathscr{F}_{n-1}\right\}, \tag{33}$$

where the contribution of $\widetilde{\mathscr{H}}_{0\,n}^{\#}$ is known and the second term contains the unknown function $\mathscr{F}_{n-1}$.

As before, by expanding the Poisson bracket on (33) and taking into account (20), the Homological equation (8) takes the form

$$\begin{aligned}\mathscr{L}_{\mathscr{H}_0}\left(\mathscr{W}_n\right) + \mathscr{K}_n &= \widetilde{\mathscr{H}}_{0\,n}^{\#} + n\sum_{i=2}^{m}\left(\frac{\partial \mathscr{H}_1}{\partial \varphi_i}\frac{\partial \mathscr{F}_{n-1}}{\partial I_i} - \frac{\partial \mathscr{H}_1}{\partial I_i}\frac{\partial \mathscr{F}_{n-1}}{\partial \varphi_i}\right).\end{aligned} \tag{34}$$

It is worth noting that this Poisson bracket can also be written as a function of the Lie operator as $-\mathscr{L}_{\mathscr{H}_1}(\mathscr{F}_{n-1})$.

Then, by substituting (33) into (17), we obtain $\mathscr{K}_n$ as

$$\begin{aligned}\mathscr{K}_n = {}& \frac{1}{2\pi}\int_0^{2\pi}\widetilde{\mathscr{H}}_{0\,n}^{\#}d\varphi_1 \\ & - n\sum_{i=2}^{m}\left(\frac{\partial \mathscr{K}_1}{\partial \varphi_i}\frac{\partial \mathscr{F}_{n-1}}{\partial I_i} - \frac{\partial \mathscr{K}_1}{\partial I_i}\frac{\partial \mathscr{F}_{n-1}}{\partial \varphi_i}\right).\end{aligned} \tag{35}$$

If we proceed in the same way as before, the first term of (35) can be decomposed again as the sum $\mathscr{K}_n^I + \mathscr{K}_n^{II}$. Then, by choosing suitable functional dependencies of $\mathscr{F}_{n-1}$, according to the additional aims of the transformation, and taking into account the identity

$$\sum_{i=2}^{m}\left(\frac{\partial \mathscr{K}_1}{\partial \varphi_i}\frac{\partial \mathscr{F}_{n-1}}{\partial I_i} - \frac{\partial \mathscr{K}_1}{\partial I_i}\frac{\partial \mathscr{F}_{n-1}}{\partial \varphi_i}\right) = \frac{1}{2n}\mathscr{K}_n^{II}, \tag{36}$$

the explicit value of $\mathscr{F}_{n-1}$ can be obtained. Again, the complexity of (36) depends on the form of $\mathscr{K}_1$ and, therefore, on the form of $\mathscr{H}_1$. Hence, the new Hamiltonian, $\mathscr{K}_n$, yields

$$\mathscr{K}_n = \mathscr{K}_n^I. \tag{37}$$

Then, by replacing the value of $\mathscr{F}_{n-1}$ into $\mathscr{W}_{n-1}$ and into the previous terms $\mathscr{H}_{i\,j}$ of the Deprit's scheme, we obtain $\mathscr{W}_n$ as

$$\mathscr{W}_n = \frac{1}{\omega}\int\left(\widetilde{\mathscr{H}}_{0\,n} - \mathscr{K}_n\right)d\varphi_1 + \mathscr{F}_n. \tag{38}$$

Finally, by taking $\mathscr{F}_n = 0$, if this function is not used to remove the terms that do not depend on φ_1 from $\mathscr{W}_n$, the transformed Hamiltonian can be expressed as

$$\mathscr{K} = \sum_{n\geq 1}\frac{\varepsilon^n}{n!}\mathscr{K}_n^I, \tag{39}$$

where the dependency on φ_1 is eliminated by the normalization process and the additional effects produced by the integration constants are included from the second order.

4. $\mathscr{K}_n$ Degenerate at Order $n \geq 1$

As has been seen in the previous section, the arbitrary integration function, when it is determined at the same order, only affects the generating function. However, if its determination is postponed, the form of the terms $\mathscr{H}_1$ and $\mathscr{K}_1$ becomes important, since it depends on the solution of the partial differential equation (36).

In this section, we assume that the first-order term $\mathscr{H}_1$ of the initial Hamiltonian (1) is transformed into a degenerated term $\mathscr{K}_1$ of the new Hamiltonian in the sense of Arnold's characterization. Under this assumption, the arbitrary function can be considered like the generator of a new Lie transform which will allow to remove any other angular variables. It is worth noting that this condition is weaker than assuming the degeneration of the term $\mathscr{H}_1$, as can be seen in the example shown in the next section.

We start considering the case in which the first-order term $\mathscr{K}_1$ depends on all the momenta I_i. It is easy to check that

$$\frac{\partial \mathscr{K}_1}{\partial \varphi_i} = 0, \quad \text{for } i = 2, \ldots, m. \tag{40}$$

On the other hand, for each $n > 1$, using the identity $\{\mathscr{H}_{1\,0}, \mathscr{F}_{n-1}\} = -\mathscr{L}_{\mathscr{H}_{1\,0}}(\mathscr{F}_{n-1})$, the Homological equation (34) can be written as

$$\mathscr{L}_{\mathscr{H}_0}\left(\mathscr{W}_n\right) + n\mathscr{L}_{\mathscr{H}_1}\left(\mathscr{F}_{n-1}\right) + \mathscr{K}_n = \widetilde{\mathscr{H}}_{0\,n}^{\#} \tag{41}$$

and, therefore, with the aid of relation (40), (35) yields

$$\mathscr{K}_n = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{\mathscr{H}}^{\#}_{0n} d\varphi_1 - n\sum_{i=2}^{m}\frac{\partial \mathscr{K}_1}{\partial I_i}\frac{\partial \mathscr{F}_{n-1}}{\partial \varphi_i}. \quad (42)$$

With the same approach used in the study of the nondegenerated case, $\mathscr{K}_n$ is taken as $\mathscr{K}^I_n$, which only depends on the momenta, and (36) becomes

$$\sum_{i=2}^{m}\frac{\partial \mathscr{K}_1}{\partial I_i}\frac{\partial \mathscr{F}_{n-1}}{\partial \varphi_i} = -\frac{1}{n}\mathscr{K}^{II}_n. \quad (43)$$

Moreover, $\mathscr{K}^{II}_n$ can be written as

$$\mathscr{K}^{II}_n = \sum_{j_1,\dots,j_m\in\vec{Z}} \mathscr{C}_{j_1,\dots,j_m}\begin{pmatrix}\sin\\ \cos\end{pmatrix}(j_2\varphi_2 + \cdots + j_m\varphi_m), \quad (44)$$

where the coefficients $\mathscr{C}_{j_2,\dots,j_m}$ are functions which depend on the momenta and the physical parameters of the initial Hamiltonian. Therefore, if we consider the nonresonant case, (43) can be solved directly and its solution is expressed as a Poisson series in the form

$$\mathscr{F}_{n-1} = \sum_{j_1,\dots,j_m\in\vec{Z}} \mathscr{C}'_{j_1,\dots,j_m}\begin{pmatrix}-\cos\\ \sin\end{pmatrix}(j_2\varphi_2 + \cdots + j_m\varphi_m), \quad (45)$$

with

$$\mathscr{C}'_{j_1,\dots,j_m} = \frac{-\mathscr{C}_{j_1,\dots,j_m}}{n(\nu_2 I_2 + \cdots + \nu_m I_m)}, \quad (46)$$

where $(\nu_2,\dots,\nu_m)$ represents the $m-1$ vector of fundamental frequencies of the first-order integrable Hamiltonian $\mathscr{K}_1$:

$$\nu_i = \frac{\partial \mathscr{K}_1}{\partial I_i}. \quad (47)$$

After that, the value of $\mathscr{F}_{n-1}$ is inserted into the previous calculations and the value of $\mathscr{W}_n$ is obtained:

$$\mathscr{W}_n = \frac{1}{\omega}\int\left(\widetilde{\mathscr{H}}_{0n} - \mathscr{K}_n\right)d\varphi_1 + \mathscr{F}_n. \quad (48)$$

The transformed Hamiltonian $\mathscr{K}$ only depends on the momenta and hence is integrable by quadratures. In this situation, the arbitrary function can be seen as the generating function of a Lie transform, which allows to remove the remaining angular variables of the transformed Hamiltonian $\mathscr{K}$. It is worth noting that the so-called double normalization algorithm [17–19] is not a different algorithm, but the classical Deprit algorithm for the particular case $\mathscr{F}_n \neq 0$.

On the other hand, an equivalent Hamiltonian, which only depends on the momenta, can also be obtained by means of two Lie transforms: the first one, to remove the variable φ_1 and then the rest of the angular variables. In both Lie transforms the arbitrary integration functions are taken equal to zero.

We would like to point out that in the case that the terms $\mathscr{H}_i$ are transformed into $\mathscr{K}_i = 0$ for $i \in \{1,\dots,p-1\}$, the Homological equation (34), which must be solved, is

$$\mathscr{L}_{\mathscr{H}_0}(\mathscr{W}_n) + \binom{n}{p}\mathscr{L}_{\mathscr{H}_p}(\mathscr{F}_{n-p}) + \mathscr{K}_n = \widetilde{\mathscr{H}}^{\#}_{0n}, \quad (49)$$

where the term $\mathscr{W}_n$ is completed at order $n+p$. This formulation can be easily extended to other similar cases. For example, if the Hamiltonian has a high-order proper degeneracy, $\mathscr{H}_1(I_{i_1}),\dots,\mathscr{H}_p(I_{i_p})$ only depend on the momenta; then the Homological equation (34), which must be solved, is

$$\mathscr{L}_{\mathscr{H}_0}(\mathscr{W}_n) + \binom{n}{p}\mathscr{L}_{\mathscr{H}_p}\left(\mathscr{F}^{i_p}_{n-p}\right) + \cdots + n\mathscr{L}_{\mathscr{H}_1}\left(\mathscr{F}^{i_1}_{n-1}\right) + \mathscr{K}_n = \widetilde{\mathscr{H}}^{\#}_{0n} \quad (50)$$

and allows removing several angular variables simultaneously, linking several transformations.

5. Application to Harmonic Oscillators

In order to provide the reader with a simple illustration of the machinery of the algorithm used by the Deprit's method in the case of $\mathscr{F}_n \neq 0$, we consider the first-order Hamiltonian system

$$\mathscr{H} = \mathscr{H}_0 + \varepsilon\mathscr{H}_1, \quad (51)$$

where $\mathscr{H}_0$ is composed of two harmonic oscillators with equal frequencies, ω,

$$\mathscr{H}_0 = \frac{1}{2}\left(X^2 + \omega x^2\right) + \frac{1}{2}\left(Y^2 + \omega y^2\right), \quad (52)$$

which is coupled by a perturbation $\mathscr{H}_1 = \delta x^2 + yx^2 + y^3$, where δ represents a small detuning parameter and ε is a small parameter. These parameters ω, δ, and ε are independent of the system variables (x, y, X, Y) which are being used. It is worth mentioning that this kind of Hamiltonian frequently appears in the context of galactic dynamics [27, 28].

First, we convert the Hamiltonian (51) into a new one of type (1); we use Lissajous variables, which were introduced by Deprit in [21] and are defined by

$$\begin{aligned} x &= s\sin(\varphi_1+\varphi_2), & X &= \omega s\cos(\varphi_1+\varphi_2),\\ y &= d\sin(\varphi_1-\varphi_2), & Y &= \omega d\cos(\varphi_1-\varphi_2), \end{aligned} \quad (53)$$

where

$$s = \sqrt{\frac{I_1+I_2}{\omega}}, \qquad d = \sqrt{\frac{I_1-I_2}{\omega}}. \quad (54)$$

In terms of Lissajous variables the unperturbed part $\mathscr{H}_0$ becomes

$$\mathscr{H}_0 = \omega I_1. \quad (55)$$

The perturbed part $\mathscr{H}_1$ takes the form

$$\mathscr{H}_1 = \frac{s^2\delta}{2} - \frac{s^2\delta}{2}\cos 2(\varphi_1+\varphi_2) + \frac{d(3d^2+2s^2)}{4}\sin(\varphi_1-\varphi_2)$$

$$-\frac{d^3}{4}\sin 3(\varphi_1-\varphi_2)-\frac{ds^2}{4}\sin(3\varphi_1+\varphi_2)$$
$$+\frac{ds^2}{4}\sin(\varphi_1+3\varphi_2), \tag{56}$$

which is nondegenerate.

In order to transform the Hamiltonian $\mathscr{H}$ into a new one independent of the angular variables up to third order, we consider two cases. First, only a Lie transform with $\mathscr{F}_n \neq 0$ is used. In the second case, two Lie transforms will be used to remove the angular variable φ_1 in first place and then φ_2, both using $\mathscr{F}_n = 0$. To conclude, we compare the composition of these two Lie transforms with the one calculated in the first case. For the sake of nomenclature simplicity, the primes from the new variables and momenta will be dropped in all cases.

5.1. Case $\mathscr{F}_n \neq 0$. Using the algorithm described in Section 3, we carry out a Lie transform so that the transformed Hamiltonian $\mathscr{K}$ is independent of the angular variables up to third order.

The method starts by taking

$$\mathscr{K}_0 = \omega I_1. \tag{57}$$

From (21) and (22) the first-order terms of the transformed Hamiltonian and generating function are given by

$$\mathscr{K}_1 = \frac{\delta s^2}{2},$$
$$\mathscr{W}_1 = -\frac{\delta s^2}{4\omega}\sin 2(\varphi_1+\varphi_2) - \frac{(2s^2+3d^2)d}{4\omega}\cos(\varphi_1-\varphi_2)$$
$$+\frac{d^3}{12\omega}\cos 3(\varphi_1-\varphi_2)+\frac{s^2 d}{12\omega}\cos(3\varphi_1+\varphi_2)$$
$$-\frac{s^2 d}{4\omega}\cos(\varphi_1+3\varphi_2)+\mathscr{F}_1, \tag{58}$$

where $\mathscr{F}_1$ depends on the variable φ_2 and the momenta I_1, I_2. This arbitrary function will be determined at the next order. It is worth noting that $\mathscr{K}_1$ is degenerate, it only depends on the momenta I_1 and I_2, although $\mathscr{H}_1$ is not.

At second order, taking into account (29), the term $\mathscr{K}_2$ has the following expression:

$$\mathscr{K}_2 = -\frac{15d^4}{8\omega^2}-\frac{11s^2d^2}{6\omega^2}-\frac{5s^4}{24\omega^2}-\frac{s^2\delta^2}{2\omega^2}$$
$$-\frac{s^2d^2}{4\omega^2}\cos 4\varphi_2-\frac{\delta}{\omega}\frac{\partial\mathscr{F}_1}{\partial\varphi_2}. \tag{59}$$

The procedure for determining the arbitrary function $\mathscr{F}_1$, as outlined by (30), takes $\mathscr{K}_2^{II}$ as the part of (59) which depends on φ_2; then it follows that the identity (30) yields

$$\frac{s^2d^2}{4\omega^2}\cos 4\varphi_2+\frac{\delta}{\omega}\frac{\partial\mathscr{F}_1}{\partial\varphi_2}=0. \tag{60}$$

Then, after some calculations, the integration constant is given by

$$\mathscr{F}_1 = -\frac{s^2d^2}{16\delta\omega}\sin 4\varphi_2. \tag{61}$$

By substituting (61) into (58), the first-order term $\mathscr{W}_1$ is completed and $\mathscr{K}_2$ yields

$$\mathscr{K}_2 = -\frac{15d^4}{8\omega^2}-\frac{11s^2d^2}{6\omega^2}-\frac{5s^4}{24\omega^2}-\frac{s^2\delta^2}{2\omega^2}, \tag{62}$$

which only depends on the momenta. The second-order term of the generating function is given by

$$\mathscr{W}_2 = \frac{5s^2d^2}{48\omega^3}\sin 4\varphi_1+\frac{(5d^2+6\delta^2+s^2)s^2}{12\delta\omega^3}\sin 2(\varphi_1+\varphi_2)$$
$$+\frac{s^4}{96\omega^3}\sin 4(\varphi_1+\varphi_2)$$
$$+\frac{(36d^2+17s^2)d^2}{48\delta\omega^3}\sin 2(\varphi_1-\varphi_2)$$
$$+\frac{3d^4}{32\omega^3}\sin 4(\varphi_1-\varphi_2)$$
$$+\frac{(3s^2-6d^2+112\delta^2)s^2d}{96\delta\omega^3}\cos(\varphi_1-\varphi_2)$$
$$+\frac{s^2d^3}{48\delta\omega^3}\cos 3(\varphi_1-\varphi_2)$$
$$+\frac{(2d^2+28\delta^2+s^2)s^2d}{16\delta\omega^3}\cos(\varphi_1+3\varphi_2)$$
$$+\frac{s^2d^3}{32\delta\omega^3}\cos(\varphi_1-5\varphi_2)$$
$$-\frac{(9d^2+8\delta^2)s^2d}{288\delta\omega^3}\cos(3\varphi_1+\varphi_2)$$
$$-\frac{s^4d}{96\delta\omega^3}\cos(3\varphi_1+5\varphi_2)+\mathscr{F}_2, \tag{63}$$

where, as a first order, $\mathscr{F}_2$ depends on the variable φ_2 and all the momenta. This arbitrary function will be determined at the next order.

At third order, taking into account (29), the term $\mathscr{K}_3$ is determined as

$$\mathscr{K}_3 = -\frac{3d^4s^2}{32\delta\omega^4}+\frac{3d^2s^4}{32\delta\omega^4}+\frac{49\delta d^2s^2}{6\omega^4}+\frac{11\delta s^4}{12\omega^4}+\frac{3\delta^3s^2}{2\omega^4}$$
$$+\frac{d^2s^2}{2\omega^4}\left(\frac{23d^2}{2\delta}+\frac{17s^2}{2\delta}+19\delta\right)\cos 4\varphi_2-\frac{3\delta}{2\omega}\frac{\partial\mathscr{F}_2}{\partial\varphi_2}. \tag{64}$$

By taking $\mathscr{K}_3^{II}$ as the part that depends on φ_2 in (64), the identity (30) reads

$$\frac{(23d^2+38\delta^2+17s^2)s^2d^2}{16\delta\omega^4}\cos 4\varphi_2-\frac{3\delta}{2\omega}\frac{\partial\mathscr{F}_2}{\partial\varphi_2}=0. \tag{65}$$

By solving (65), the integration constant is given by

$$\mathcal{F}_2 = \frac{\left(23d^2 + 38\delta^2 + 17s^2\right)s^2d^2}{96\delta^2\omega^3}\sin 4\varphi_2. \tag{66}$$

By substituting (66) into (63), the second-order term $\mathcal{W}_2$ is completed and $\mathcal{K}_3$ yields

$$\mathcal{K}_3 = -\frac{3d^4s^2}{32\delta\omega^4} + \frac{3d^2s^4}{32\delta\omega^4} + \frac{49\delta d^2s^2}{6\omega^4} + \frac{11\delta s^4}{12\omega^4} + \frac{3\delta^3s^2}{2\omega^4}, \tag{67}$$

which, similarly, only depends on the momenta and, hence, is trivially integrable. The value of $\mathcal{W}_3$ is

$$\begin{aligned}
\mathcal{W}_3 = &-\frac{\left(33d^2 + 27s^2 + 700\delta^2\right)s^2d^2}{576\delta\omega^5}\sin 4\varphi_1 \\
&+ \mathcal{P}_1 \sin 2\left(\varphi_1 + \varphi_2\right) \\
&- \frac{\left(21d^2 + 6s^2 + 166\delta^2\right)s^4}{576\delta\omega^5}\sin 4\left(\varphi_1 + \varphi_2\right) \\
&+ \frac{s^6}{1152\delta\omega^5}\sin 6\left(\varphi_1 + \varphi_2\right) \\
&+ \frac{\left(156d^2 + 99s^2 - 716\delta^2\right)s^2d^2}{288\delta\omega^5}\sin 2\left(\varphi_1 - \varphi_2\right) \\
&- \frac{s^2d^4}{64\delta\omega^5}\sin 4\left(\varphi_1 - \varphi_2\right) \\
&- \mathcal{P}_2\cos\left(\varphi_1 - \varphi_2\right) + \mathcal{P}_3\cos 3\left(\varphi_1 - \varphi_2\right) \\
&+ \frac{27d^5}{80\omega^5}\cos 5\left(\varphi_1 - \varphi_2\right) \\
&- \frac{41s^4d^2}{192\delta\omega^5}\sin 2\left(\varphi_1 + 3\varphi_2\right) - \mathcal{P}_4\cos\left(\varphi_1 + 3\varphi_2\right) \\
&- \frac{47s^2d^4}{384\delta\omega^5}\sin 2\left(\varphi_1 - 3\varphi_2\right) \\
&+ \frac{\left(23d^2 + 12s^2 - 100\delta^2\right)s^2d^3}{192\delta^2\omega^5}\cos\left(\varphi_1 - 5\varphi_2\right) \\
&- \frac{11s^4d^3}{192\delta^2\omega^5}\cos\left(\varphi_1 + 7\varphi_2\right) + \frac{s^4d^2}{192\delta\omega^5}\sin 2\left(3\varphi_1 + \varphi_2\right) \\
&+ \left(621d^4 + \left(306s^2 + 2408\delta^2\right)d^2\right. \\
&\qquad \left. + 4\delta^2\left(89s^2 - 48\delta^2\right)\right)\frac{s^2d}{1728\delta^2\omega^5}\cos\left(3\varphi_1 + \varphi_2\right) \\
&+ \frac{\left(103d^2 + 34s^2 + 92\delta^2\right)s^4d}{576\delta^2\omega^5}\cos\left(3\varphi_1 + 5\varphi_2\right) \\
&- \frac{23s^2d^5}{192\delta^2\omega^5}\cos\left(3\varphi_1 - 7\varphi_2\right) \\
&+ \frac{97s^2d^3}{240\omega^5}\cos\left(5\varphi_1 - \varphi_2\right) \\
&+ \frac{7s^4d}{80\omega^5}\cos\left(5\varphi_1 + 3\varphi_2\right) + \mathcal{F}_3,
\end{aligned} \tag{68}$$

with

$$\begin{aligned}
\mathcal{P}_1 &= \frac{s^2}{1152\delta\omega^5}\left(-18d^4 + 64\left(3s^2 - 35\delta^2\right)d^2 + 21s^4\right. \\
&\qquad \left. - 2304\delta^4 - 208s^2\delta^2\right), \\
\mathcal{P}_2 &= \frac{d}{576\delta^2\omega^5}\left(102s^6 + 3120\delta^2s^4 + 4480\delta^4s^2\right. \\
&\qquad - 12d^4\left(23s^2 - 945\delta^2\right) \\
&\qquad \left. + d^2\left(11508s^2\delta^2 - 99s^4\right)\right), \\
\mathcal{P}_3 &= \frac{d^3}{576\delta^2\omega^5}\left(-125s^4 + 140\delta^2s^2 + d^2\left(720\delta^2 - 92s^2\right)\right), \\
\mathcal{P}_4 &= \frac{s^2d}{192\delta^2\omega^5}\left(253d^4 + \left(274s^2 + 960\delta^2\right)d^2\right. \\
&\qquad \left. + 4\left(17s^4 + 67\delta^2s^2 + 688\delta^4\right)\right).
\end{aligned} \tag{69}$$

Finally, by taking $\mathcal{F}_3 = 0$, the third-order theory is completed. It is worth noting that the process takes place within the Poisson algebra.

In this example, the performed Lie transform, taking into account the $\mathcal{F}_n \neq 0$ case in Deprit's method, removes both angular variables (φ_1, φ_2) at once. Then, the Hamilton's equation associated with the transformed Hamiltonian $\mathcal{K}$ can be written as

$$\begin{aligned}
\dot{\varphi}_1 &= \omega + \frac{\delta}{2\omega}\varepsilon - \frac{6\delta^2 + 27s^2 + 67d^2}{24\omega^3}\varepsilon^2 \\
&\quad + \frac{-9d^4 + 144\delta^4 + 784\delta^2d^2 + 9s^4 + 960\delta^2s^2}{576\delta\omega^5}\varepsilon^3, \\
\dot{\varphi}_2 &= \frac{\delta}{2\omega}\varepsilon - \frac{6\delta^2 - 17s^2 - 23d^2}{24\omega^3}\varepsilon^2 \\
&\quad + \frac{-9d^4 + 144\delta^4 + 4d^2\left(196\delta^2 + 9s^2\right) - 9s^4 - 608\delta^2s^2}{576\delta\omega^5}\varepsilon^3, \\
\dot{I}_1 &= 0, \\
\dot{I}_2 &= 0,
\end{aligned} \tag{70}$$

which are easily integrated by quadratures.

5.2. Case $\mathcal{F}_n = 0$. Using the classical choice of $\mathcal{F}_n$ in Deprit's algorithm, we present here the transformed Hamiltonians and the generating functions of two Lie transforms. The first Lie transform develops so as to remove the angular variable φ_1 and the second transform does the same with the other angular variable φ_2, so that the final Hamiltonian is independent of all the angular variables up to third order.

The first transformed Hamiltonian through third order in ε is

$$\mathscr{K}' = \mathscr{K}'_0 + \varepsilon\mathscr{K}'_1 + \frac{\varepsilon^2}{2}\mathscr{K}'_2 + \frac{\varepsilon^3}{3!}\mathscr{K}'_3, \tag{71}$$

where

$$\begin{aligned}
\mathscr{K}'_0 &= \omega I,\\
\mathscr{K}'_1 &= \frac{\delta s^2}{2},\\
\mathscr{K}'_2 &= -\frac{15d^4}{8\omega^2} - \frac{11d^2s^2}{6\omega^2} - \frac{5s^4}{24\omega^2} - \frac{\delta^2 s^2}{2\omega^2} - \frac{d^2s^2}{4\omega^2}\cos 4\varphi_2,\\
\mathscr{K}'_3 &= \frac{49d^2\delta s^2}{6\omega^4} + \frac{11\delta s^4}{12\omega^4} + \frac{3\delta^3 s^2}{2\omega^4} + \frac{11d^2\delta s^2}{4\omega^4}\cos 4\varphi_2,
\end{aligned} \tag{72}$$

whereas the generating function yields

$$\mathscr{U} = \mathscr{U}_1 + \varepsilon\mathscr{U}_2 + \frac{\varepsilon^2}{2}\mathscr{U}_3, \tag{73}$$

with

$$\begin{aligned}
\mathscr{U}_1 = {} & -\frac{\delta s^2}{4\omega}\sin 2(\varphi_1+\varphi_2) - \frac{(3d^3+2ds^2)}{4\omega}\cos(\varphi_1-\varphi_2)\\
& + \frac{d^3}{12\omega}\cos 3(\varphi_1-\varphi_2) + \frac{ds^2}{12\omega}\cos(3\varphi_1+\varphi_2)\\
& - \frac{ds^2}{4\omega}\cos(\varphi_1+3\varphi_2),\\
\mathscr{U}_2 = {} & \frac{5d^2s^2}{48\omega^3}\sin 4\varphi_1 + \frac{s^2(5d^2+6\delta^2+s^2)}{12\omega^3}\sin 2(\varphi_1+\varphi_2)\\
& + \frac{s^4}{96\omega^3}\sin 4(\varphi_1+\varphi_2)\\
& + \frac{(9d^4+5d^2s^2)}{12\omega^3}\sin 2(\varphi_1-\varphi_2)\\
& + \frac{3d^4}{32\omega^3}\sin 4(\varphi_1-\varphi_2) + \frac{7\delta ds^2}{6\omega^3}\cos(\varphi_1-\varphi_2)\\
& - \frac{\delta ds^2}{36\omega^3}\cos(3\varphi_1+\varphi_2) + \frac{7\delta ds^2}{4\omega^3}\cos(\varphi_1+3\varphi_2),\\
\mathscr{U}_3 = {} & -\frac{359\delta d^2s^2}{288\omega^5}\sin 4\varphi_1\\
& - \frac{(140d^2+144\delta^2+13s^2)\delta s^2}{72\omega^5}\sin 2(\varphi_1+\varphi_2)\\
& - \frac{83\delta s^4}{288\omega^5}\sin 4(\varphi_1+\varphi_2) - \frac{245\delta d^2s^2}{72\omega^5}\sin 2(\varphi_1-\varphi_2)\\
& - \frac{(2835d^4+3018d^2s^2+669s^4+1120\delta^2s^2)d}{144\omega^5}\\
& \cdot\cos(\varphi_1-\varphi_2)\\
& + \frac{45d^5+23d^3s^2}{36\omega^5}\cos 3(\varphi_1-\varphi_2)\\
& + \frac{27d^5}{80\omega^5}\cos 5(\varphi_1-\varphi_2)\\
& - \frac{(173d^2+688\delta^2+22s^2)ds^2}{48\omega^5}\cos(\varphi_1+3\varphi_2)\\
& - \frac{11d^3s^2}{48\omega^5}\cos(\varphi_1-5\varphi_2)\\
& + \frac{(101d^2-12\delta^2+29s^2)ds^2}{108\omega^5}\cos(3\varphi_1+\varphi_2)\\
& + \frac{ds^4}{18\omega^5}\cos(3\varphi_1+5\varphi_2)\\
& + \frac{17d^3s^2}{40\omega^5}\cos(5\varphi_1-\varphi_2) + \frac{7ds^4}{80\omega^5}\cos(5\varphi_1+3\varphi_2).
\end{aligned} \tag{74}$$

Then, the variable φ_2 is removed using a second Lie transform. The transformed Hamiltonian through third order in ε is

$$\mathscr{K}'' = \mathscr{K}''_0 + \varepsilon\mathscr{K}''_1 + \frac{\varepsilon^2}{2}\mathscr{K}''_2 + \frac{\varepsilon^3}{3!}\mathscr{K}''_3, \tag{75}$$

where

$$\begin{aligned}
\mathscr{K}''_0 &= \omega I,\\
\mathscr{K}''_1 &= \frac{\delta s^2}{2},\\
\mathscr{K}''_2 &= -\frac{15d^4}{8\omega^2} - \frac{11d^2s^2}{6\omega^2} - \frac{5s^4}{24\omega^2} - \frac{\delta^2s^2}{2\omega^2},\\
\mathscr{K}''_3 &= -\frac{3d^4s^2}{32\delta\omega^4} + \frac{3d^2s^4}{32\delta\omega^4} + \frac{49\delta d^2s^2}{6\omega^4} + \frac{11\delta s^4}{12\omega^4} + \frac{3\delta^3s^2}{2\omega^4},
\end{aligned} \tag{76}$$

whereas the generating function is given by

$$\mathscr{V} = \mathscr{V}_1 + \varepsilon\mathscr{V}_2 + \frac{\varepsilon^2}{2}\mathscr{V}_3, \tag{77}$$

with

$$\begin{aligned}
\mathscr{V}_1 &= -\frac{d^2s^2}{16\delta\omega}\sin 4\varphi_2,\\
\mathscr{V}_2 &= \frac{(23d^2+38\delta^2+17s^2)d^2s^2}{96\delta^2\omega^3}\sin 4\varphi_2,\\
\mathscr{V}_3 &= 0.
\end{aligned} \tag{78}$$

Note that, up to third order, the Hamiltonian $\mathscr{K}$, obtained by Deprit's method considering $\mathscr{F}_n \neq 0$, is identical to the Hamiltonian $\mathscr{K}''$, which has been obtained after applying two Lie transforms, generated, respectively, by $\mathscr{U}$ and $\mathscr{V}$, and taking the arbitrary integration constant equal to zero. On the other hand, in order to relate the generating functions $\mathscr{W}$, $\mathscr{U}$, and $\mathscr{V}$, we consider the composition of the two previous Lie transforms taking into account the following identity:

$$\mathscr{T} = \mathscr{U} + \mathscr{L}_{\mathscr{U}}^{-1}(\mathscr{V}). \tag{79}$$

Then, we consider the difference between $\mathscr{W}$ and $\mathscr{T}$ up to third order. It is easy to check that $\mathscr{W}_1 - \mathscr{T}_1 = \mathscr{W}_2 - \mathscr{T}_2 = 0$, whereas at third order we obtain that

$$\mathscr{W}_3 - \mathscr{T}_3 = \frac{\left(-19d^2 - 9\delta^2 + s^2\right) d^2 s^2}{288\delta\omega^5} \sin 4\varphi_2, \tag{80}$$

which is a function that belongs to $\ker(\mathscr{L}_{\mathscr{H}_0})$.

6. Conclusion

Deprit's method has been revisited so as to clarify and identify the role played by the intrinsic arbitrariness which appears in all perturbation techniques. This arbitrariness arises as an arbitrary integration function, when the generating function of the Lie transform is calculated from the Homological equation in Deprit's method. This function belongs to the kernel of the Lie transform. In the case of a degenerate Hamiltonian in Arnold's sense, this function allows removing the implicit terms belonging to the kernel of the Lie transform embedded in the generating function or can be used as the generator of a new transformation through which other angular variables can be removed. Moreover, other algorithms derived from Deprit's method, like the so-called double normalization, are put in the correct context and their relation with classical Deprit's method clarified. Finally, exactly the same reasoning can be applied to other techniques like Hori's method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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