Estimation of the lot fraction defective in a finite lot of products with auxiliary quality characteristics

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ABSTRACT
Many companies use sampling plans for the acceptance or rejection of lots of products. The final outcome of this decision-making process is based on the inspection of a sample of products selected from a lot under inspection, where a quality characteristic is observed. An important parameter of interest related to acceptance sampling for attributes is the proportion of defective items. This parameter is unknown for a given lot of products, but it can be estimated from the aforementioned sample information. Additional quality characteristics can be observed at the inspection stage. We propose to use this auxiliary information to obtain more accurate estimators of the lot fraction defective at the estimation stage. Various relevant applications of this process are described. For possible scenarios that may arise in practice, the empirical properties of the suggested estimation methods are investigated using Monte Carlo simulations, and desirable results are obtained when there is a strong relationship between the quality characteristic of interest and the auxiliary quality characteristic.

KEYWORDS
quality control; operating characteristic curve; lot fraction defective; distribution function; Monte Carlo simulation

1. Introduction

In today's competitive markets, companies have to provide products and services with the best possible relationship between quality and price. In this respect, the use of quantitative techniques for assessing and monitoring the quality of business production plays an important role in achieving product improvements and boosting productivity. This in turn may enhance customer satisfaction, drive sales and thus increase earnings. The set of statistical tools used to assess the quality of products is referred to as statistical quality control, and it can be divided into three categories (Besterfield 2014; Mitra 2008; Montgomery 2009): statistical process control; acceptance sampling; and design of experiments.

Acceptance sampling is one of the oldest quality assurance techniques, and it involves the inspection of and decision-making about products, i.e., acceptance sampling is used to decide whether to accept or reject lots of products shipped from producers/suppliers (Baklizi and El Masri 2004). We refer to the process of inspecting the sample of products as the inspection stage. Acceptance sampling can be performed using single, double or multiple sampling plans (Balakrishnan, Leiva, and López 2007; Montgomery 2009; Tomohiro, Arizono, and Takemoto 2016). A sampling plan basically consists of a sample size and the decision criteria for acceptance or rejection of the lot under inspection. This paper discusses the acceptance sampling for attributes based on a single sampling plan. The quality of a lot of products under inspection is analyzed in relation to one or more product characteristics, called quality characteristics. A product is said to be defective if at least one of the examined quality characteristics does not satisfy a specific quality requirement. For example, a product can be classified as defective if the quality characteristic is outside the specification limits, which are usually set by customer requirements (Montgomery 2009; Muñoz-Rosas et al. 2016).

An important parameter related to a single sampling plan for attributes is the proportion of defective items. For a given lot submitted for inspection, this parameter is defined as

$$p = \frac{D}{N}, \tag{1}$$

where N is the lot size,

$$D = \sum_{i=1}^{N} d_i \tag{2}$$

is the number of defective items in the lot, and d_i takes the value 1 if the ith item in the lot is considered as defective, and $d_i = 0$ otherwise. Note that p is unknown at the lot level, since it depends on D, which is also unknown. However, p may be estimated by using the sample information, and this phase is referred to as the estimation stage. This information on the proportion of defective items has multiple applications, hence the estimation of p with desirable properties has a relevant interest in practice. The main purpose of this paper is to investigate different methods of estimating the parameter p, and to determine which of these may provide more accurate results than the customary method used in practice and defined in Section 2 (see equation (3)).

It is fairly common practice to analyze various quality characteristics of the products under inspection when conducting acceptance sampling. For example, multivariate quality control (Cabana and Lillo 2022; Lowry and Montgomery 1995) assesses several related quality characteristics. For various reasons (to avoid multicollinearity, for the sake of simplicity, etc.), univariate quality control can be a better option than multivariate quality control. This issue is discussed in Section 2.1. We consider univariate quality control, but the information collected from the additional quality characteristics is taken into account at the estimation stage (Berger and Muñoz 2015; Fernández and Pérez-González 2012) to obtain a more accurate estimator of the proportion of defective items (p).

The estimation of our parameter of interest p is based on difference type estimators (Rao, Kovar, and Mantel 1990), but alternative estimation methods can be found in the literature, such as poststratification (Silva and Skinner 1995) and logistic regression (Lehtonen and Veijanen 1998) estimators.

Our results indicate that the proposed estimator is more efficient than the customary method, when there is a strong relationship between the auxiliary quality characteristic and the quality characteristic of interest. In particular, the proposed method can be more efficient when the Pearson's correlation coefficient (ρ) is larger than 0.9, and the gain in efficiency increases as ρ increases. Note that the presence of a strong relationship between quality characteristics is quite common in several products, especially when such quality characteristics are based on continuous measurements. For instance, certain specification limits are required of the products obtained from a production process, and for stable processes (i.e., in statistical control) with small variations and cases where the use of correlations is appropriate (e.g., quality characteristics based on continuous measurements), it is reasonable to assume a strong correlation between the quality characteristics. A number of studies support this argument. For example, Caudill et al. (1992) simulate various quality characteristics to investigate the effect of different correlation coefficients, and the case of $\rho = 0.9$ is included. Shil, Singh, and Mehta (2019) and Roy et al. (2021) use multivariate statistical quality control techniques, with many quality characteristics showing values of ρ larger than 0.9. Liu et al. (2018) identify a quality characteristic of interest, and the value of ρ with an auxiliary quality characteristic is $\rho = 0.871$.

The rest of the paper is organized as follows. Some applications of the problem of estimating the proportion of defective items are described in Section 1.1. The customary estimator of p is described in Section 2, along with a discussion of different estimation methods based on auxiliary information and applied to the context of acceptance sampling for attributes. The empirical properties of the various estimation methods are investigated in Section 3 using Monte Carlo simulations, showing desirable results under certain conditions. Our findings are summarized in Section 4.

1.1. *Justification for the estimation of p and applications*

Although acceptance sampling was not designed for estimation purposes, the information observed in this process can be valuable for many reasons. First, both producers and consumers want to avoid costly mistakes in the phase of accepting or rejecting a lot (Heizer and Render 2014), i.e., the producer wants to avoid the mistake of having a good lot rejected, as it typically has to be replaced (this is referred to as the producer risk), and the customer wants to avoid the mistake of accepting a bad lot, since defects found in a lot that has already been accepted are usually the consumer's responsibility (this is referred to as the consumer risk). Note that many sampling plans are based on both producer and consumer risks (see, for example, Aslam et al. 2013; Divecha and Raykundaliya 2022; Kannan, Jeyadurga, and Balamurali 2022). A well-designed acceptance sampling plan not only reduces the cost and time of inspection but also provides the desired protection to the producer and the consumer (Wu et al. 2015). An estimated value of the proportion of defective items may provide both producer and consumer with valuable information about the actual level of quality of the lot under inspection, and this resulting knowledge is often a useful input into the overall quality planning and engineering process.

Furthermore, it is generally accepted that lots should be homogeneous, and if this is not the case, the consumer may put financial or psychological pressure on the supplier to improve the production process. The proportion of defective items can be an appropriate indicator in this situation. For instance, an unusual value of the proportion of defective items can motivate the supplier to revise and/or improve the process control.

Various curves based upon the proportion of defective items are commonly used to evaluate sampling plans. The operating characteristic (OC) curve (Mitra 2008; Montgomery 2009) analyzes the performance of the acceptance sampling plan, i.e., it graphs the discriminatory power of the sampling plan. The OC curve plots the values of the theoretical probability of accepting a lot (P_a) versus different values of p. The average outgoing quality (Duffuaa and Khan 2008; Mitra 2008; Montgomery 2009), AOQ, is a measure used for the evaluation of a rectifying sampling, i.e., it gives the expected average level of quality of the items in a lot when it leaves an inspection point. The AOQ is defined as

$$AOQ = \frac{P_a p(N-n)}{N},$$

where n is the sample size associated with the sampling plan. The AOQ curve plots the values of AOQ versus different values of p. Finally, the average total inspection (Mitra 2008; Montgomery 2009), ATI, gives the average number of items inspected per lot, and this measure is defined as

$$ATI = n + (1 - P_a)(N - n).$$

Likewise, the ATI curve plots the values of ATI versus different values of p. We observe that the parameter p is used in the construction of the OC, AOQ and ATI curves, which in turn are used to evaluate sampling plans. An estimation of p can be used to detect possible deviations between this quantity and the theoretical value of p. In such cases, the aforementioned deviation between p and its estimator may have an impact on the evaluation of sampling plans. In addition, the calculation of P_a depends on the proportion of defective items p (Montgomery 2009, 638), which may be unknown and then must be replaced by an accurate estimate of this parameter for the problem of estimating the probability of accepting a lot. For the reasons given above, good estimators of p are required in the context of sampling plans.

The consumer, sometimes in conjunction with the producer through contractual agreements, may specify two target values in a given sampling plan: the acceptable quality level (AQL) and the rejectable quality level (RQL). Sampling plans can be designed to specify performance at the AQL and RQL values, and the proportion of defective items in the lot (p) plays an important role in this performance (Chukhrova and Johannssen 2018). For instance, the consumer quite often designs the sampling plan so that the probability of acceptance in the OC curve is large at the AQL, and the correct evaluation of the OC curve depends on an efficient estimation of p. For this purpose, an accurate estimation of the proportion of defective items is essential, and it can be used as an audit tool to ensure that the output of a process meets requirements (Montgomery 2012). Similarly, the knowledge of the proportion of defective items is key for sampling plans based on the Process Capability Index (PCI), as can be seen in Pearn and Wu (2007), Wu and Pearn (2008), and Yen et al. (2020).

Sampling plans are not static: each sampling plan can be replaced by another one that is better suited to possible new conditions and/or requirements. In addition, sampling plans that require much less inspection can be implemented as the producer/supplier builds a satisfactory relationship with the consumer over time, and this desirable reputation of quality can be supported by the results of the sampling activities. Indeed, the estimation of the proportion of defective items can be used for this purpose.

2. Estimation based on auxiliary quality characteristics

The acceptance sampling technique is required because the value of D is unknown, which implies that the parameter p is also unknown according to equation (1). The value of p may be estimated by using the information collected from the sampling plan. We consider a single sampling plan for attributes (see Montgomery 2009), which consists in selecting a sample s, with size n, from the lot with size N. Then, the observed number of defective items in the sample,

$$d = \sum_{i=1}^{n} d_i,$$

is compared to the acceptance number c, which is set in advance. In other words, the lot under inspection is accepted if $d \leq c$, and it is rejected otherwise. The customary estimator of the lot fraction defective p when samples are selected under simple random sampling without replacement is given by

$$\widehat{p} = \frac{d}{n}. \tag{3}$$

2.1. *Auxiliary quality characteristics*

Let y be the quality characteristic of interest. The observed value of y for the ith product in the lot is denoted by y_i, where $i = \{1, \ldots, N\}$.

Many databases are obtained from the lot inspection stage, and they may contain several quality characteristics that are related to the quality characteristic of interest (Amiri, Zou, and Doroudyan 2014). Some examples of these characteristics can be found in studies on multivariate quality control, as it is a method based on several related quality characteristics (Lowry and Montgomery 1995; Shahriari and Abdollahzadeh 2009). For instance, when controlling a fiber-production process, the single-strand break factor can be evaluated through an auxiliary quality characteristic such as the weight of the textile fibers. Also, in the manufacturing process for a specific carbon fiber tubing where the quality characteristic of interest is the inner diameter, two related quality characteristics usually considered are the thickness and the length of tubes (Haq 2017), and this justifies their use as auxiliary characteristics. This information is sometimes available because many of the quality characteristics are based on a simple measurement that is not difficult to obtain and track. Although the multivariate methods can offer important benefits, they can also suffer from certain problems, such as the fact that they cannot be applied in the presence of multicollinearity, which is more likely to appear when the relationship between the quality characteristics is strong. In addition, the univariate methods may perform better than the multivariate methods, and in such case the univariate techniques are preferred because they are easier to use and simpler to interpret than multivariate methods.

We consider the univariate case, i.e., we assume processes based on a single quality characteristic. However, databases may contain additional quality characteristics related to the characteristic of interest. For instance, products derived from a production process may have to meet some fixed specifications, and it is reasonable to assume that some of the quality characteristics have a strong correlation when the process is stable and variations are small. This is the case with the examples illustrated by Shil, Singh, and Mehta (2019) and Roy et al. (2021), where the quality characteristics have values of ρ larger than 0.9. The Pearson's correlation coefficient between the auxiliary quality characteristic and quality characteristic of interest in the example described by Liu et al. (2018) is $\rho = 0.871$. The idea of this paper is to define the parameter p (see equation (1)) in terms of the distribution function, and then make use of the auxiliary quality characteristics at the estimation stage, since the proper use of the auxiliary information may provide estimators with a smaller variance or mean square error (Berger and Muñoz 2015; Muñoz, Álvarez-Verdejo, and García-Fernández 2018; Rao, Kovar, and Mantel 1990). For the variable y and the observations taken from a lot, with size N, the distribution function for a given argument t is defined as:

$$F(t) = \frac{1}{N}\sum_{i=1}^{N}\delta(y_i \le t), \tag{4}$$

where $\delta(A)$ is the indicator variable that takes the value 1 if the expression A is true, and the value 0 otherwise. For instance, $\delta(y_i \le t) = 1$ if $y_i \le t$, and $\delta(y_i \le t) = 0$ if $y_i > t$. Thus, $F(t)$ gives the percentage of products in the lot satisfying $y_i \le t$. For simplicity, we assume a single auxiliary quality characteristic, which is denoted by x. However, the proposed estimators can easily be extended to several auxiliary quality characteristics (see Rao, Kovar, and Mantel 1990), which could yield more accurate results.

2.2. *Suggested estimation methods*

In this section, the lot fraction defective p is expressed in terms of the distribution function $F(\cdot)$ defined by equation (4) (see equation 5). Thus, estimators of p can be proposed by substituting each distribution function $F(\cdot)$ with estimators of this parameter that are based on auxiliary information, and which may have a smaller mean square error. For simplicity, we assume specifications limits, which are values between which the quality characteristic should operate in order to have a non-defective product. For the case of a process with two-sided specification limits, the ith product is classified as defective if the quality characteristic is outside the lower (LSL) and the upper (USL) specification limits, i.e., $y_i < LSL$ or $y_i > USL$. The values d_i (see equation (2)) are given by

$$d_i = \delta(y_i < LSL) + \delta(y_i > USL),$$

where $d_i = 1$ indicates that the ith product is defective, and $d_i = 0$ otherwise. The extension to one-sided specification limits is straightforward. The lot fraction defective can be defined as

$$p = \frac{1}{N}\sum_{i=1}^{N} d_i = \frac{1}{N}\sum_{i=1}^{N}[\delta(y_i < LSL) + \delta(y_i > USL)] =$$

$$= \frac{1}{N}\sum_{i=1}^{N}[\delta(y_i \le LSL) + 1 - \delta(y_i \le USL) - \delta(y_i = LSL)] =$$

$$= 1 + F(LSL) - F(USL) - P(LSL), \tag{5}$$

where

$$P(t) = \frac{\delta(y_i = t)}{N}$$

is the proportion of items in the lot that satisfy the condition $y_i = t$. Note that $P(LSL)$ approaches 0 when y is a continuous quality characteristic.

For data sets with auxiliary quality characteristics related to the quality characteristic of interest, the estimation stage may include such auxiliary information, i.e., estimators based on auxiliary variables can be used. The related literature suggests that estimation methods based on auxiliary variables are expected to produce more accurate results (Muñoz, Álvarez-Verdejo, and García-Fernández 2018; Rao, Kovar, and Mantel 1990). In the presence of auxiliary information, difference type estimators can be used to improve the estimation of a given parameter. We investigate the empirical properties of two difference type estimators in the context of acceptance sampling plans. Such estimators are denoted by $\widehat{p}_d$ and $\widehat{p}_{d.dm}$ in this paper, and they are defined using the notation followed by Rao, Kovar, and Mantel (1990). The classical difference type estimator of $F(t)$ is given by

$$\widehat{F}_d(t) = \widehat{F}(t) + F_x(t) - \widehat{F}_x(t),$$

where

$$\widehat{F}(t) = \frac{1}{n}\sum_{i=1}^{n} \delta(y_i \leq t),$$

$$\widehat{F}_x(t) = \frac{1}{n}\sum_{i=1}^{n} \delta(\widehat{\beta} x_i \leq t)$$

and

$$F_x(t) = \frac{1}{N}\sum_{i=1}^{N} \delta(\widehat{\beta} x_i \leq t),$$

where

$$\widehat{\beta} = \frac{\sum_{i=1}^{n} y_i x_i / \upsilon^2(x_i)}{\sum_{i=1}^{n} x_i^2 / \upsilon^2(x_i)}$$

is the estimator of the parameter β under the regression model

$$y_i = \beta x_i + \upsilon(x_i) u_i \quad (i = 1, \ldots, N), \tag{6}$$

and where $\upsilon(\cdot)$ is a known, strictly positive function, and the u_i are independent and identically distributed random variables with mean equal to 0. The use of $\upsilon(\cdot)$ in this regression model indicates that this model has heteroscedastic errors (Berger and Muñoz 2015). It is fairly standard practice to consider $\upsilon(x_i) = x_i^{1/2}$ (Muñoz, Álvarez-Verdejo, and García-Fernández 2018), which implies that

$$\widehat{\beta} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i}$$

in this situation. Similarly, $P(t)$ may be estimated by using a difference type estimator. For this purpose, we replace $\delta(y_i \leq t)$ by $\delta(y_i = t)$ and $\delta(\widehat{\beta}x_i \leq t)$ by $\delta(\widehat{\beta}x_i = t)$, i.e.,

$$\widehat{P}_d(t) = \widehat{P}(t) + P_x(t) - \widehat{P}_x(t),$$

where

$$\widehat{P}(t) = \frac{1}{n}\sum_{i=1}^{n} \delta(y_i = t),$$

$$\widehat{P}_x(t) = \frac{1}{n}\sum_{i=1}^{n} \delta(\widehat{\beta}x_i = t)$$

and

$$P_x(t) = \frac{1}{N}\sum_{i=1}^{N} \delta(\widehat{\beta}x_i = t).$$

The difference type estimator of p is defined as

$$\widehat{p}_d = \max\{0, 1 + \widehat{F}_d(LSL) - \widehat{F}_d(USL) - \widehat{P}_d(LSL)\}. \tag{7}$$

Note that $\widehat{p}_d$ is also motivated by the results derived from Rao, Kovar, and Mantel (1990), which justify the fact that the values

$$\widehat{d_i} = \delta(\widehat{\beta}x_i < LSL) + \delta(\widehat{\beta}x_i > USL) = \delta(x_i < LSL_x) + \delta(x_i > USL_x)$$

are predictors of the values d_i in the regression equation (6). This issue can be used to extend this method to cases involving several auxiliary quality characteristics. For this purpose, such auxiliary variables must be included in the regression model. The value $\widehat{d_i} = 1$ indicates that the ith item is defective according to the estimation based on x and model (6), and where the specification limits for the auxiliary quality characteristic are given by

$$LSL_x = \frac{LSL}{\widehat{\beta}}; \quad USL_x = \frac{USL}{\widehat{\beta}}.$$

Assuming this second estimation perspective, the difference type estimator can also be defined as

$$\widehat{p}_d = \max\{0, \widehat{p}_d^*\},$$

where

$$\widehat{p}_d^* = \widehat{p} + p_x - \widehat{p}_x,$$

$\widehat{p}$ is defined in equation (3), and

$$\widehat{p}_x = \frac{1}{n} \sum_{i=1}^{n} \widehat{d}_i$$

is the customary estimator of

$$p_x = \frac{1}{N} \sum_{i=1}^{N} \widehat{d}_i.$$

The lot fraction defective p commonly takes values close to 0. If this is indeed the case, the estimator $\widehat{p}_d^*$ can take values smaller than 0, since the difference estimator $\widehat{F}_d(t)$ proposed by Rao, Kovar, and Mantel (1990) can take values outside the interval $[0, 1]$. The estimator $\widehat{p}_d$ is thus truncated at 0 to avoid the presence of negative values. Note that this technique is usually applied when an estimation method can provide results outside a reasonable range of values. For example, limits of confidence intervals for proportions are truncated when any of them take values outside the interval [0,1]. Rao, Kovar, and Mantel (1990) report some desirable properties related to the estimator $\widehat{F}_d(t)$ of the distribution function $F(t)$, and they can also be applied to the context of acceptance sampling for attributes. For instance, there is a relevant gain in the efficiency of the estimator $\widehat{p}_d$ when the quality characteristic y is approximately proportional to the quality characteristic x.

Rao, Kovar, and Mantel (1990) also defined a difference type estimator of $F(t)$, which has the desirable property of being asymptotically both design-unbiased and model-unbiased under model (6). This estimator of the distribution function is given by

$$\widehat{F}_{d.dm}(t) = \widehat{F}(t) + \frac{1}{N} \sum_{i=1}^{N} \widehat{G}_i - \frac{1}{n} \sum_{i=1}^{n} \widehat{G}_i,$$

where

$$\widehat{G}_i = \frac{1}{n} \sum_{j=1}^{n} \delta \left(\widehat{u}_j \leq \frac{t - \widehat{\beta} x_i}{\nu(x_i)} \right)$$

and

$$\widehat{u}_j = \frac{y_j - \widehat{\beta} x_j}{\nu(x_j)}.$$

The proportion $P(t)$ can also be estimated using this method, i.e.,

$$\widehat{P}_{d.dm}(t) = \widehat{P}(t) + \frac{1}{N}\sum_{i=1}^{N}\widehat{G}_i^* - \frac{1}{n}\sum_{i=1}^{n}\widehat{G}_i^*,$$

where

$$\widehat{G}_i^* = \frac{1}{n}\sum_{j=1}^{n}\delta\left(\widehat{u}_j = \frac{t-\widehat{\beta}x_i}{\nu(x_i)}\right).$$

Following the definition of p given by equation (5), the second difference type estimator of p is defined as

$$\widehat{p}_{d.dm} = \max\{0, 1 + \widehat{F}_{d.dm}(LSL) - \widehat{F}_{d.dm}(USL) - \widehat{P}_{d.dm}(LSL)\}.$$

We observe that the auxiliary quality characteristic needs to have all the measurements corresponding to the lot. This requirement for estimators based on auxiliary information has been widely discussed in the literature. First, auxiliary quality characteristics must be chosen in such a way that they are based on a simple measurement that is not difficult to obtain and track. In the case of a lot of products, the weight of the product can be a good candidate for an auxiliary quality characteristic, since the total weight of the lot, for example, can be easily measured. Alternatively, different statistical methods have been proposed in the literature to estimate the total sums associated with auxiliary variables. For instance, a simple solution is to use two-phase sampling (Legg and Fuller 2009; Muñoz, Álvarez, and Rueda 2014). Finally, the extended regression estimator (Berger, Muñoz, and Rancourt 2009) can also be used. This estimation method is frequently used in business surveys, as well as in official surveys such as the Canadian Labour Force Survey. Särndal, Swensson, and Wretman (2003) state that the use of estimators based on auxiliary variables can give more accurate results, and they can be easily applied to a variety of practical situations.

3. Monte Carlo simulations

In this section we investigate the empirical properties of the various estimators of p using Monte Carlo simulation (Jurun and Pivac 2011; Silva and Skinner 1995). For this purpose, we use real and artificial data sets, which are treated as finite lots of products, to which single sampling plans are applied.

3.1. *Description of lots and sampling plans*

We generated artificial lots to represent some real cases that can be observed in practice. The effects of various estimation methods under such scenarios are then analyzed. For example, we empirically investigate the impact on the estimation stage of different correlation coefficients reflecting the relation between the auxiliary quality characteristic and the quality characteristic of interest. Moreover, we investigate the efficiency in relation to the problem of estimating the proportion of defective items in the data set pistonrings (Montgomery 2009; Scrucca 2004). The effect of the value of ρ on the estimation stage is also analyzed by simulating auxiliary quality characteristics with different correlations. Note that this method was also used by Caudill et al. (1992). The main characteristics of the different finite lots and the sampling plans used in this study are described below. The OC curves related to the sampling plans used in this study are also presented.

First, we consider finite lots with sizes $N = \{500, 1000, 10000\}$, and set the specification limits at $LSL = 95$ and $USL = 105$. For each lot size N, we generate the quality characteristics of interest

$$y_j \to N(\mu_y, \sigma_{yj}),$$

where μ_y is the target value, σ_{yj} is the standard deviation, and j denotes the proportion of defective items associated with the quality characteristic y_j when the specification limits $LSL = 95$ and $USL = 105$ are used. We consider a proportion of defective items between 1% and 10% ($p = \{0.01, 0.02, \ldots, 0.10\}$) to investigate the impact of different values of p on the various estimation methods. We choose the parameters $\mu_y = 100$ and

$$\sigma_{yj} = \frac{5}{z_{1-j/2}}$$

because they provide our desired proportions of defective items. In particular, we find that the selected values for μ_y, σ_{yj}, LSL and USL satisfy

$$p = 1 - P(LSL \leq y_j \leq USL) = 1 - P\left(\frac{LSL - \mu_y}{\sigma_{yj}} \leq Z \leq \frac{USL - \mu_y}{\sigma_{yj}}\right) =$$

$$= 1 - P(-z_{1-j/2} \leq Z \leq z_{1-j/2}) = 1 - (1 - j) = j,$$

which justifies j coinciding with the proportion of defective items, and explains the choice for the parameters μ_y, σ_{yj}, LSL and USL.

For each quality characteristic of interest y_j, an auxiliary quality characteristic x_{jk} is obtained using the equation

$$x_{jk} = y_j + \epsilon_{xk},$$

where

$$\epsilon_{xk} \to N(0, \sigma_{\epsilon k}).$$

The standard deviation $\sigma_{\epsilon k}$ is selected such that the correlation coefficients between the auxiliary quality characteristic and quality characteristic of interest take the values $\rho = \{0.7, 0.9, 0.95\}$, and the performance of the analyzed estimation methods under different correlation coefficients can thus be investigated. We consider the sampling plans $\{n = 50; c = 4\}$ and $\{n = 100; c = 8\}$ in the case of artificial lots. The OC curves related to the various sampling plans are plotted in Figure 1.

We also consider a finite lot based on the pistonrings data set (Montgomery 2009; Scrucca 2004), where the quality characteristic y is the inside diameter of pistons used in an automobile transmission. This data set is duplicated twice to get a finite lot

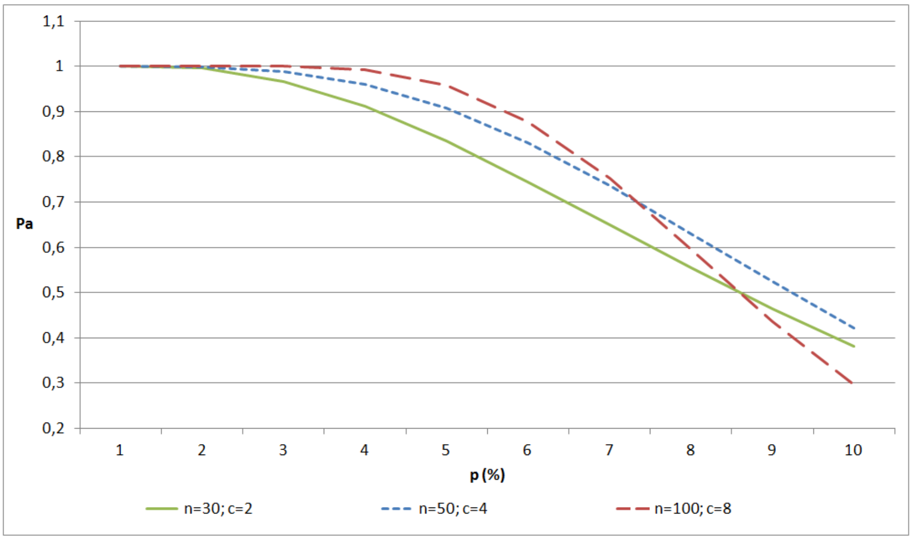

Figure 1. OC curves for various sampling plans used in the Monte Carlo simulations. The sampling plans $\{n = 50; c = 4\}$ and $\{n = 100; c = 8\}$ are used in the case of artificial lots, and the sampling plan $\{n = 30; c = 2\}$ in the pistonrings data set.

Table 1. Lower and Upper specification limits (LSL and USL, respectively) and proportions of defective items considered in the pistonrings data set.

LSL	USL	p
73.9780	74.0220	0.02
73.9815	74.0185	0.05
73.9845	74.0155	0.11

with $N = 250$ pistons. The mean and the standard deviation of y are given by 74 and 0.01 millimeters, respectively. The considered specification limits and values of p are described in Table 1.

Following Caudill et al. (1992), the quality characteristic x_k is obtained using the procedure described for artificial lots, i.e.,

$$x_k = y/2 + \epsilon_{xk},$$

where

$$\epsilon_{xk} \to N(0, \sigma_{\epsilon k}).$$

We consider the values $\sigma_{\epsilon k} = \{0.0025, 0.0015\}$ for the standard deviation, since they provide, respectively, the values $\rho = \{0.9, 0.95\}$ for the correlation coefficients between the auxiliary quality characteristic and quality characteristic of interest. The sampling plan $\{n = 30; c = 2\}$ is considered in this case, and the corresponding OC curve is also plotted in Figure 1.

The aim of this paper is to evaluate the performance of estimators of p. The value of p has an important effect on the performance of estimators, hence the suggested Monte Carlo simulation is based on a wide range of values for this parameter. In addition, we evaluate additional aspects that have a relevant impact on the estimation, such as the correlation coefficient, the lot size, and the sample size. We consider $n = \{30, 50, 100\}$

to investigate the performance of estimators under small and medium sample sizes. Larger sample sizes are omitted because it is expected that the various estimators all perform well in this situation. Summarizing, this study is based on different values of N, n, c, p, and ρ, and it implies that a total of 186 different scenarios are analyzed. Data sets and the open-source code (using the statistical software R) used to obtain the results derived from simulation studies are available in Muñoz et al. (2023).

3.2. *Description of empirical measures for the comparison of estimators*

The most common measures used to compare the precision of different estimation methods are the empirical relative root mean square error ($RRMSE$) and the empirical relative bias (RB). The various estimators of p are compared in terms of $RRMSE$ and RB, which are defined as

$$RRMSE[\widetilde{p}] = 100 \times \frac{\sqrt{MSE[\widetilde{p}]}}{p},$$

and

$$RB[\widetilde{p}] = 100 \times \frac{E[\widetilde{p}] - p}{p}$$

where $\widetilde{p}$ denotes a given estimator of the parameter p, and $E[\cdot]$ and $MSE[\cdot]$ are, respectively, the empirical expectation and the empirical mean square error based on $R = 10000$ simulation runs, i.e.,

$$E[\widetilde{p}] = \frac{1}{R}\sum_{i=1}^{R}\widetilde{p}(i)$$

and

$$MSE[\widetilde{p}] = \frac{1}{R}\sum_{i=1}^{R}(\widetilde{p}(i) - p)^2,$$

where $\widetilde{p}(i)$ denotes the value of the estimator $\widetilde{p}$ at the ith simulation run. Other authors that use the same empirical measures are Muñoz, Álvarez-Verdejo, and García-Fernández (2018), and Silva and Skinner (1995).

3.3. *Results and conclusions*

For the case of artificial lots, the performance of the various estimators of p can be seen in Figures 2 and 3. Figure 2 analyzes the efficiency of the various estimators in terms of the empirical measure $RRMSE$. The estimator $\widehat{p}_{d.dm}$ is more efficient than $\widehat{p}_d$, since $\widehat{p}_{d.dm}$ has the smallest values of $RRMSE$ regardless of p values. As we expected, the difference type estimators of p ($\widehat{p}_d$ and $\widehat{p}_{d.dm}$) become more efficient as the linear correlation coefficient (ρ) increases. In particular, they are more efficient than the customary estimator ($\widehat{p}$) when $\rho = \{0.9, 0.95\}$. We also observe that the various estimators have a better performance in terms of efficiency as values of both

n and p increase. For the extreme case of low proportions, the estimator $\widehat{p}_{d.dm}$ also performs better than $\widehat{p}$ under the scenario of a strong relationship between the quality characteristics ($\rho = 0.95$).

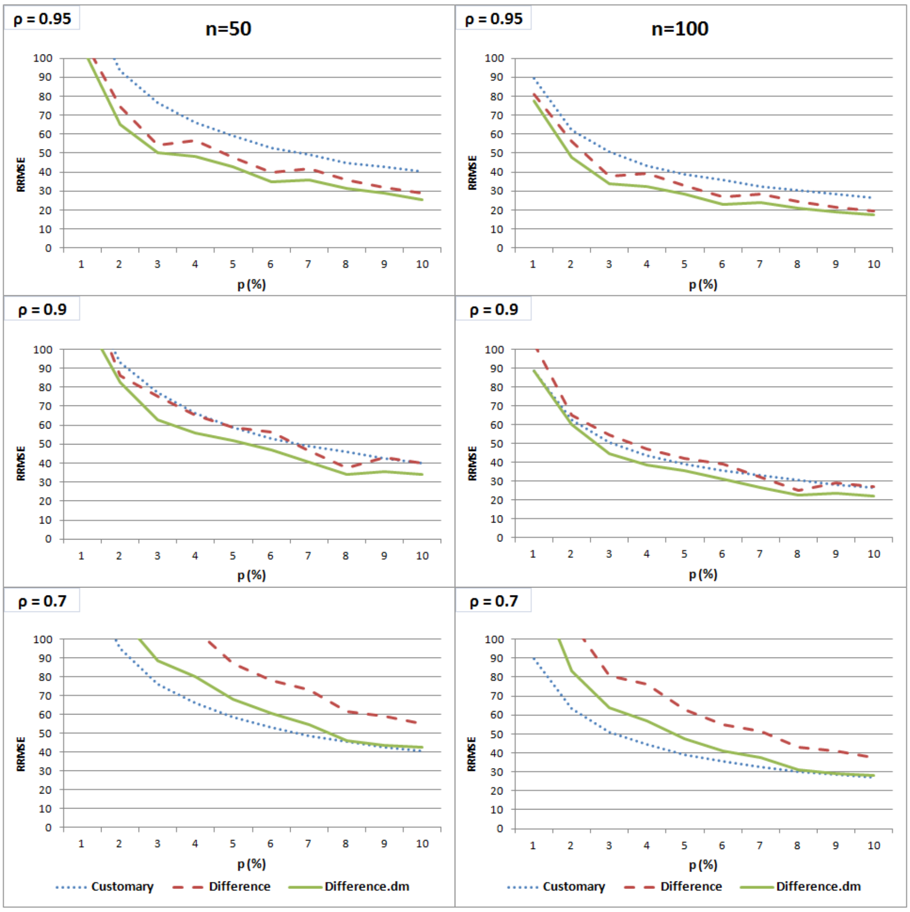

Figure 2. Values of the empirical measure $RRMSE$ of the various estimators of the proportion of defective items (p): customary ($\widehat{p}$); difference ($\widehat{p}_d$); and difference.dm ($\widehat{p}_{d.dm}$). We considered finite lots with sizes $N = 500$, the sampling plans $\{n = 50; c = 4\}$ and $\{n = 100; c = 8\}$, and linear correlation coefficients $\rho = \{0.7, 0.9, 0.95\}$.

As far as the bias of the various estimators is concerned (see Figure 3), we observe that the customary estimator gives values of RB close to 0% for the various values of p. The biases of the difference type estimators are smaller as the values of p increase. For instance, the estimator $\widehat{p}_{d.dm}$ has reasonable biases, with values of RB less than 5%, when p is larger than 1% and $\rho = \{0.9, 0.95\}$. The bias of $\widehat{p}_{d.dm}$ under the extreme case of low proportions ($p = 1\%$) can be explained by the fact that this estimator is truncated at 0, as discussed in Section 2. However, we observe that this issue does not have an impact on the efficiency of $\widehat{p}_{d.dm}$, since this estimator is more efficient than $\widehat{p}$ when $p = 1\%$. Moreover, we observe that the biases of $\widehat{p}_{d.dm}$ are negligible when $p = 1\%$ if we increase the sample size (n) and/or ρ is large. The empirical properties of the different estimators of p were also investigated for finite lots with sizes $N = \{1000, 10000\}$; since similar conclusions were reached under the new scenarios, the results are omitted.

Results derived from the finite lots with sizes $N = 500$ (Figures 2–3) indicate that the estimator $\widehat{p}_{d.dm}$ is more efficient than the customary estimator $\widehat{p}$ for large

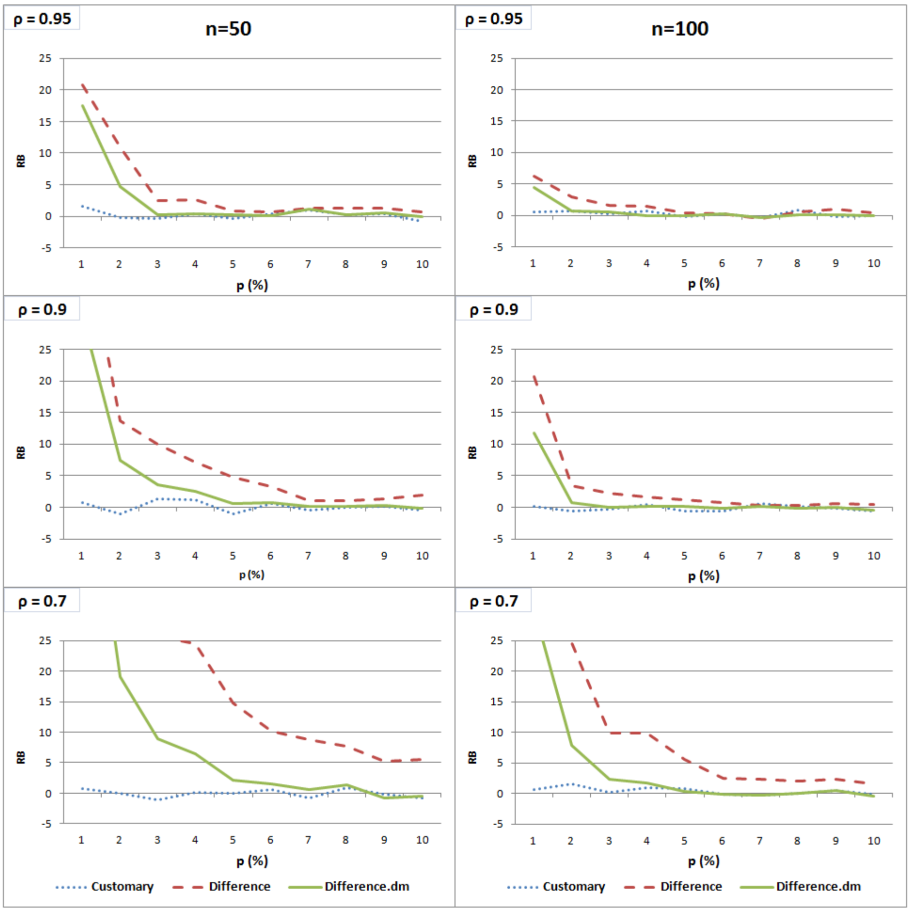

Figure 3. Values of the empirical measure RB of the various estimators of the proportion of defective items (p): customary ($\widehat{p}$); difference ($\widehat{p}_d$); and difference.dm ($\widehat{p}_{d.dm}$). We considered finite lots with sizes $N = 500$, the sampling plans $\{n = 50; c = 4\}$ and $\{n = 100; c = 8\}$, and linear correlation coefficients $\rho = \{0.7, 0.9, 0.95\}$.

Pearson's correlation coefficients. The same conclusion can be drawn from Table 2, which contains the empirical results derived from the pistonrings data set. $\widehat{p}_{d.dm}$ is more efficient than $\widehat{p}$ for the various cases analyzed in this table, although the gain in efficiency is more notable when the proportion of defective items is small and the linear correlation coefficient is large. For example, the values of $RRMSE$ of the estimators $\widehat{p}_{d.dm}$ and $\widehat{p}$ are, respectively, 44.8 and 110.7 when $p = 0.02$ and $\rho = 0.95$. In general, reasonable biases are observed for the various estimators.

4. Discussion and concluding remarks

Acceptance sampling plans are decision-making tools used in many companies for the acceptance or rejection of finite lots of products. Some tools that can be used for the evaluation of sampling plans are the OC, AOQ and ATI curves. What these curves have in common is the fact that they depend on the proportion of defective items (p). In addition, sampling plans may be based on target values, such as the AQL and the RQL, and the parameter p also plays an important role in this situation. In practice, the value of p associated with a given sampling plan is fixed, but its value is unknown for a given lot under inspection, and it may differ from the fixed target value. This

Table 2. Values of the empirical measures $RRMSE$ and RB of the various estimators of the proportion of defective items (p): customary ($\widehat{p}$); difference ($\widehat{p}_d$); and difference.dm ($\widehat{p}_{d.dm}$). We considered the pistonrings data set ($N = 250$), the sampling plan $\{n = 30; c = 2\}$, and linear correlation coefficients $\rho = \{0.9, 0.95\}$.

		$RRMSE$			RB		
ρ	p	$\widehat{p}$	$\widehat{p}_d$	$\widehat{p}_{d.dm}$	$\widehat{p}$	$\widehat{p}_d$	$\widehat{p}_{d.dm}$
0.95	0.02	110.7	38.7	44.8	0.2	5.6	0.6
	0.05	76.3	55.2	50.7	0.8	4.3	0.8
	0.11	48.9	38.2	34.7	0.5	1.1	0.2
0.90	0.02	109.2	38.5	44.9	-0.8	5.8	0.5
	0.05	76.2	54.8	49.9	0.5	5.4	1.5
	0.11	48.7	38.6	34.8	0.7	0.5	0.0

is one reason why accurate estimators of p are needed. Random samples are used in the process of implementing acceptance sampling plans, and the quality characteristic of interest is observed at this stage. Sample information can be used to estimate the parameter p. In addition, the sample may contain information on additional quality characteristics, which could be used at the estimation stage to obtain more accurate estimators of p.

The main aim of this paper is to investigate this idea and analyze the empirical properties of the suggested estimation methods based on auxiliary quality characteristics. First, we showed that the parameter p can be expressed in terms of the distribution function $F(\cdot)$, which is defined in equation (4). Consequently, p is estimated using estimators of the distribution function. We considered difference type estimators, since they are reliable methods in the context of survey sampling. Note that alternative estimators of the distribution function have also been analyzed, such as the poststratification (Silva and Skinner 1995) and the logistic regression (Lehtonen and Veijanen 1998) estimators. However, the suggested difference type estimators performed better at estimating p, and for this reason the alternative estimation methods are omitted.

The assumption of the existence of one or more auxiliary quality characteristics is justified by the extensive literature on multivariate quality control (Cabana and Lillo 2022; Lowry and Montgomery 1995), i.e., it is fairly common to assume that the sample may contain information on various quality characteristics. The novelty of this paper is to implement acceptance sampling plans based on a single quality characteristic, but making good use of the additional information at the estimation stage. This methodology is a popular topic in the context of survey sampling (Berger and Muñoz 2015; Rao, Kovar, and Mantel 1990; Särndal, Swensson, and Wretman 2003), and may yield desirable results in terms of accuracy.

The empirical properties of the various estimators have been analyzed under different scenarios, i.e., we considered various types of lots, sampling plans, correlation coefficients between the quality characteristics, etc. First, we observed that the difference type estimator $\widehat{p}_{d.dm}$ is the most efficient (in terms of $RRMSE$) estimator when the value of the linear correlation coefficient between the quality characteristics is large ($\rho = \{0.9, 0.95\}$), and an important gain in efficiency has been observed in the pistonrings data set. The difference type estimator $\widehat{p}_d$ performs better than the corresponding customary estimator under a strong linear correlation coefficient ($\rho = 0.95$). For $\rho = \{0.9, 0.95\}$, the estimator $\widehat{p}_{d.dm}$ has reasonable relative biases when p is larger than 1%. For $p = 1\%$, the biases of $\widehat{p}_{d.dm}$ decrease as the sample size (n) and/or the linear correlation coefficient (ρ) increase.

For simplicity, we considered the standard linear regression model defined by equation (6). We expect an improvement in the results if the proposed model had a better fit to the data under study. However, the idea was to describe the suggested estimation methods using a simple and common linear model. The extension to alternative models is straightforward and it may yield more accurate results. Likewise, we considered a single auxiliary quality characteristic at the estimation stage. The suggested estimation methods can also be generalized to the case of several auxiliary quality characteristics (Särndal, Swensson, and Wretman 2003), and desirable results are also expected.

Data availability statement

Data sets and all the analyses are open and reproducible. We used a Rmarkdown document to explain statistical computing concepts and provide the R scripts used in this article. This supplemental material for reproducibility can be seen at the OSF repository (see the file "186DataSetsRcodesProportions.html" in Muñoz et al. (2023)).

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